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Team from United Kingdom

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Problem Number 12 – Helmholtz carousel

Attach Christmas tree balls on a low friction mounting (carousel) such that the hole in each ball points in a tangential direction. If you expose this arrangement to sound of a suitable frequency and intensity, the carousel starts to rotate. Explain this phenomenon and investigate the parameters that result in the maximum rotation speed of the carousel.
Introduction

The Helmholtz* Resonator is a roughly spherical glass or metal vessel with a wide opening at one end and a narrow opening at the other.

*After Hermann Ludwig Ferdinand von Helmholtz (1821 – 1894)
Introduction

- A Helmholtz resonator is described by the following parameters:

- Cross-sectional area of neck $= A$
- Neck length $= L$
- Volume $= V$
Derivation of Resonant Frequency

Force on the column of air in the neck of the resonator due to a pressure change $dP$ is

$$ F = AdP = A \frac{dP}{dV} \Delta V = AV \frac{dP}{dV} \frac{Ax}{V} $$

For an ideal gas undergoing adiabatic expansion/contraction we have adiabatic bulk modulus

$$ K_s = -V \frac{dP}{dV} = \gamma P_0 $$

$$ F = ma = -\frac{A^2 \gamma P_0}{V} x = \rho AL \frac{d^2 x}{dt^2} $$

$$ \frac{d^2 x}{dt^2} + \frac{A \gamma P_0}{V \rho L} x = 0 $$

Which is the equation describing SHM with:

$$ \omega_0 = \sqrt{\frac{A \gamma P_0}{V \rho L}} = v \sqrt{\frac{A}{VL}} $$

Where $v$ is the speed of sound, and we have

$$ f_0 = \frac{v}{2\pi} \sqrt{\frac{A}{VL}} $$
Introduction

• The resonant frequency of a Helmholtz resonator can be found by blowing across the top of the neck. (We used a compressed air supply.)

235.8Hz
Introduction

• A simple calculation for the bottle gives a resonant frequency of 310Hz, which is too high compared to the measured 235.8Hz.

• Using 235.8Hz gives a neck length of 34mm which is 14mm longer than the measured value.

• This suggests an end correction of 14mm.
Introduction

• The bottle resonates, but this does not explain how it generates a force that might propel the carousel

• A simple experiment demonstrates the existence of the force

• An increase in ‘mass’ of 0.05g suggests thrust = 0.0005±0.0001N
Introduction

- A more sensitive balance (to 0.001g) allowed us to investigate the dependence of the force on frequency.
- The peak at 227Hz is consistent with previous findings.
The Carousel

- A pair of the previous bottles were used to make a crude carousel.

- The carousel propels itself around when the bottles resonate (at 227Hz).
The Carousel

- The problem asks for Christmas tree balls so another carousel was made.

- This propels itself around nicely at a resonant frequency of 321Hz.
The Carousel

The previous experiments suggest that:

• Rotation of the carousel will only happen if the resonators are resonating.
• Rotation speed depends of the loudness of the sound
• Friction is minimal, so very low rotation speeds are possible
• Air drag is the significant retarding force.
Where does the driving force come from?

• At first glance it seems to be a mystery!
• If air flow in and out of neck is symmetrical in every respect, there is no net change in momentum and no net force.
• If, let’s say, air flows out quickly but is drawn in relatively slowly, the situation is different.

\[ F = \nu \frac{dm}{dt} \]

• As the bulk modulus of air is dependent on pressure, this may well account for the effect.
**Force produced by asymmetry**

Adiabatic process so \( \frac{P}{P_0} = \left( \frac{V_0}{V} \right)^\gamma \) and \( \frac{P + \Delta P}{P_0} = \left( \frac{V_0}{V_0 - \Delta V} \right)^\gamma \) so \( \Delta P = P_0 \left( \frac{V_0}{V_0 - \Delta V} \right)^\gamma - P_0 \)

Force on air in neck is \( F = A\Delta P = AP_0 \left( \left( \frac{V_0}{V_0 - \Delta V} \right)^\gamma - 1 \right) = AP_0 \left( \left( 1 - \frac{\Delta V}{V_0} \right)^{-\gamma} - 1 \right) \)

Binomial expansion approximates this to \( F = AP_0 \left( \gamma \frac{\Delta V}{V_0} + \frac{\gamma(\gamma + 1)}{2} \left( \frac{\Delta V}{V_0} \right)^2 \right) \)

If we take the value of \( \gamma \) to be 1.4 for air, we have \( F = AP_0 \left( 1.4 \frac{\Delta V}{V_0} + 1.7 \left( \frac{\Delta V}{V_0} \right)^2 \right) \)

If the volume is constrained to vary sinusoidally, we obtain this plot.

This is for a rather exaggerated \( \frac{\Delta V}{V_0} = 0.1 \) but it demonstrates the principle.
Momentum considerations

- Air flows out – this air gains momentum to the right
- Some momentum transferred to air not originally in bottle
- The bottle gains momentum to the left

- Air flows in – this air gains momentum to the left
- Crucially, the horizontal component lost by the surrounding air is now less than above.
- The bottle gains momentum to the right.
- There is a net gain in momentum to the left for the bottle each cycle = \( \Delta p_i - \Delta p_o \).
Factors affecting speed

- Sound level affects rotational speed of the carousel. It will not rotate at all at low sound levels.
Factors affecting speed

- The speed is affected by the force and hence the parameters of the resonator that determine the force.
- We varied neck length and area of cross-section.

- Reducing the neck diameter from 15mm to 9mm reduced the resonant frequency from 226.2Hz to 141.6Hz.
- This is in line with the expectations.
- The force was reduced from 0.00128N to 0.00011N.
Factors affecting speed

- The force and hence the speed is a maximum at the resonant frequency of the Helmholtz oscillators, in this case 305.4Hz
Factors affecting speed

- During the acceleration part of the carousel’s motion, speed depends on time as well as force.

- The graph is parabolic - suggesting constant acceleration.
Factors affecting speed

• A graph of angle against $t^2$ should give a straight line.

• The gradient = 0.446 degrees s$^{-2}$ or 0.00778 rad s$^{-2}$

• The mean angular acceleration is therefore 0.0156 rad s$^{-2}$ over the first 16s
Torque

- The previous angular acceleration can give a torque value as $\Gamma = I\alpha$
- $I = 0.000328\text{kgm}^2$ and $\alpha = 0.0156\text{rad s}^{-2}$
- Torque = 0.00000512Nm
- Or thrust per ball = 0.0000142N
- Measured thrust per ball was 0.00041N @ 305.4Hz which is considerably greater
- Difference due to distance of balls from loudspeakers and balls not being in optimum position
Conclusion/summary

• The ‘Helmholtz Carousel’ uses four suitably oriented Helmholtz resonators.

• Helmholtz resonators generate an average thrust over each expansion and contraction cycle due to the dependence of bulk modulus on pressure leading to asymmetry in the flow velocity into and out of the resonator.

• The speed of rotation of the carousel depends on:
  • Excitation sound frequency
  • Excitation sound amplitude
  • Geometric parameters of each resonator e.g. neck length and diameter

Bulk Modulus (of elasticity)

• The **bulk modulus**, \( K = -V \frac{dP}{dV} \) or \( \rho \frac{dP}{d\rho} \)

• In relation to thermodynamic properties, \( K_S = \gamma P \) for adiabatic processes and \( K_T = P \) for isothermal processes

• \( \gamma \) is the **heat capacity ratio** or **adiabatic index** or **ratio of specific heats**, is the ratio of the heat capacity at constant pressure \( C_P \) to heat capacity at constant volume \( C_V \).

• Rapid expansion and contraction is an adiabatic process and is governed by the relationship \( PV^\gamma = constant \) whereas isothermal processes are governed by \( PV = constant \), i.e. Boyle’s law.

• The speed of sound is given by the Newton-Laplace equation \( c = \sqrt{\frac{K}{\rho}} \)

• \( c = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{N_A m}} = \sqrt{\frac{\gamma RT}{M}} \)
Moment of Inertia of Carousel

- Moment of Inertia of thin spherical shell = \( \frac{2}{3} Mr^2 \)

- By parallel axis theorem, Moment of Inertia of a thin spherical shell about axis of carousel = \( \frac{2}{3} Mr^2 + MR^2 \)

- Moment of Inertia of a rod rotated about its centre = \( \frac{1}{3} MR^2 \)

- Moment of Inertia of Carousel = \( 4 \left( \frac{2}{3} M_B r^2 + M_B R^2 \right) + 2 \left( \frac{1}{3} M_S R^2 \right) = 0.000328 \text{ kgm}^2 \)

- Ignores contribution of pin and glue